Int. J. Heat Mass Transfer. Vol. 23, pp. 1521–1530
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0017-9310/80/1101-1521 $02/00/0

BUOYANCY INDUCED TRANSPORT IN POROUS MEDIA SATURATED WITH PURE OR SALINE WATER AT LOW TEMPERATURES

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(*Received* 7 *May* 1979 *and in revised form* 31 *January* 1980)

Abstract — Terrestrial water is commonly found at temperature levels at which a density extremum, or a trend towards an extremum, occurs. The buoyancy force, with an extremum, then may become very complicated, with local flow reversal and convective inversion. Density differences may then not be expressed linearly in temperature. Such flows are formulated here, calculating the buoyancy force from a density equation of state which is of very high accuracy, yet of simple form. The resulting equations of motion are applied to vertical plane flows in a porous medium. Several new parameters arise. However, a similarity formulation results for many interesting and practical applications, for a wide diversity of temperature conditions. Specific solutions are given for flow and transport for bounding temperature conditions at which local buoyancy force reversal as well as convective inversion occur.

NOMENCLATURE

b, similarity transformation function for independent variable, defined in equation (12);
C, similarity transformation function for stream function, defined in equation (13);
$C_1, \ldots C_4$, arbitrary constants defined in equations (20, 21);
$C_5, \ldots C_{11}$, arbitrary constants of integration;
c_p, specific heat;
d, wall-to-environment temperature function defined in equation (15b);
f, similarity stream function variable;
g, acceleration of gravity;
j, environment temperature function defined in equation (15a);
k, effective thermal conductivity of the saturated porous medium;
K, permeability of porous material;
m, exponent, in equation (35a);
M, constant in equation (35a);
n, power-law exponent in equation (27a);
N, constant in equation (27a);
Nu_x, local Nusselt number;
p, pressure;
q, exponent in the density equation (4);
Q, energy of convected fluid in the boundary layer;
R, parameter defined in equation (22);
Ra_x, local Rayleigh number defined in equation (28);
s, salinity;
t, temperature;
u, Darcy velocity in x direction;
v, Darcy velocity in y direction;
V, vector velocity;
x, coordinate parallel to surface;
y, coordinate normal to surface.

Greek letters

α, coefficient in the density equation (4);
α_1, thermal-diffusivity ratio of matrix conductivity to fluid heat capacity;
δ, boundary-layer thickness;
η, independent similarity variable;
μ, viscosity of fluid;
ρ, density;
ϕ, normalized temperature;
ψ, stream function.

Subscripts

c, denotes point where flow reversal occurs;
f, refers to convective fluid;
m, denotes quantities at extremum temperature;
r, denotes reference quantity;
x, means differentiation with respect to x;
∞, denotes conditions at infinity.

1. INTRODUCTION

THE BUOYANCY induced motion of water through permeable material is an important mechanism of energy transport. In addition to considerable experimental work [1–5], there have been many theoretical studies as well [6–11]. Minkowycz and Cheng [12], Cheng and Minkowycz [13], and Johnson and Cheng [14] have obtained a number of similarity solutions for free convection boundary-layer flow. In all past analyses, the Boussinesq approximation, that the fluid density ρ varies linearly with temperature, is invoked. However, this is inapplicable for water at low temperatures. Recall the extremum at about 4°C in pure water at 1 atm. Such conditions occur commonly in porous medium, such as permeable soils flooded by cold lake or sea water, water–ice slurries, etc. Sun and Tien [15] have considered the convection of water as a non-Boussinesq fluid with a cubic polynomial density–temperature relationship. This study is a linear stability analysis. The results were restricted to conditions wherein the extremum temperature lies within the bounding temperatures, i.e. $t_1 < t_m < t_2$.

The present paper uses both a more accurate and simpler density equation which applies to both pure and saline water to a pressure level of 1000 bars, to 20°C. It is used to study vertical buoyancy driven plane flows imbedded in an extensive porous medium saturated with either pure or saline water under conditions in which density extremum might occur. The necessary and sufficient conditions for similarity solutions are determined. We take a Cartesian coordinate system with the origin at the leading edge of the flow, as shown in Fig. 1, where x increases in the downstream direction.

Some simplifying hypotheses have been made: the saturating liquid and the porous layer are in local thermodynamic equilibrium; the physical properties of the fluid and the medium are isotropic and homogeneous; the empirical Darcy's law is assumed; there is no salinity diffusion, and the Dufour and Sorét effects are both taken as negligible, for small wall-to-ambient temperature differences.

With these assumptions, the governing equations are given by:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\mathbf{V} = \frac{K}{\mu}(\rho \mathbf{g} - \nabla p), \tag{2}$$

$$(\rho_r c_p)_f \mathbf{V} \cdot \nabla t = k \nabla^2 t, \tag{3}$$

$$\rho = \rho_m(s,p)[1 - \alpha(s,p)\,|t - t_m(s,p)|^q] \tag{4}$$

where $\mathbf{V}$ is the vector of Darcy velocity, μ and c_p are the viscosity and specific heat of the convective fluid and ρ_r is its density at a reference temperature. Also, K and k are, respectively, the permeability and the effective thermal conductivity of the saturated porous medium [16, 17], p and $\mathbf{g}$ are the fluid pressure and the gravitational acceleration, s is the salinity level of the water.

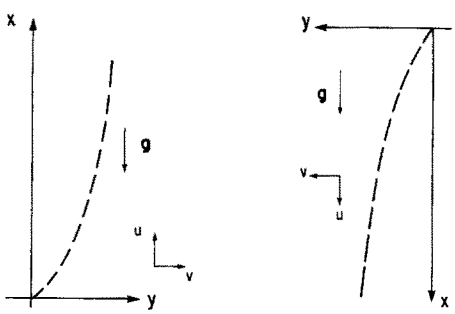

FIG. 1. Coordinate systems for the two flow regimes: (a) upward flow, (b) downward flow.

In writing equation (1), one Boussinesq approximation resulting from $\Delta\rho/\rho \ll 1$ is employed. However, the buoyancy force in equation (2) is calculated from the new density equation (4), where ρ_m and t_m denote the maximum density and temperature for given pressure and salinity levels. The forms and values of q, α, ρ_m and t_m are given in all detail by Gebhart and Mollendorf [18].

2. ANALYSIS

For a vertical plane flow in a saturated porous medium, u and v are the downstream and normal components of the filtration velocity $\mathbf{V}$. Darcy's law (2) and the equation of the energy (3) are written as

$$u = \frac{K}{\mu}\left(\pm \rho g - \frac{\partial p}{\partial x}\right), \qquad v = -\frac{K}{\mu}\frac{\partial p}{\partial y}, \tag{5a, b}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{(\rho_r c_p)_f}\left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right] \tag{6}$$

where the plus sign in equation (5a) is for the coordinate system shown in Fig. 1(b) and the minus sign is for that in Fig. 1(a). For an impermeable surface at $y = 0$ with a prescribed temperature, the appropriate boundary conditions are:

at the surface:

$$y = 0, \quad v = 0, \quad t = t_0; \tag{7a, b}$$

far from the surface, in a quiescent and stably stratified ambient medium:

$$y \to \infty, \quad u = 0, \quad t = t_\infty. \tag{8a, b}$$

For other flows, such as in a plume, other conditions apply at $y = 0$.

The pressure terms appearing in equation (5) are eliminated through cross-differentiation. Then the boundary-layer approximations in the study of free convection in porous layers are applied by assuming that the convection takes place within a thin layer adjacent to $y = 0$. They result in neglecting the changes of physical quantities, with respect to x, compared to those with respect to y. With these considerations, equations (5) and (6) become

$$\frac{\partial u}{\partial y} = \pm\frac{K}{\mu} g \frac{\partial \rho}{\partial y}, \tag{9}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{(\rho_r c_p)_f}\frac{\partial^2 t}{\partial y^2} = \alpha_1 \frac{\partial^2 t}{\partial y^2}. \tag{10}$$

Integrating equation (9) with respect to y and applying the boundary condition (8a), results in

$$u(x,y) = \pm\frac{K}{\mu} g(\rho_\infty - \rho),$$

or using the density equation of state (4)

$$u(x,y) = \pm \frac{K\alpha(s,p)\rho_m q}{\mu}[|t - t_m|^q - |t_\infty - t_m|^q] \quad (11)$$

where the effect of pressure variation on density has been neglected and salinity is taken as uniform. This result indicates that the velocity u is directly proportional to the buoyancy force. Therefore, as shown in Fig. 3, in terms of the eventual similarity variables, the vertical velocity and the buoyancy force variations are of the same shape. Where local buoyancy force reversal occurs, flow reversal occurs (Fig. 5). Furthermore, if equation (11) is applied at the surface, there is slip, since velocity u varies in the same manner as the buoyancy force.

Equations (7a, b), (8b), (10) and (11) are the governing boundary layer equations and boundary conditions for buoyancy induced convection in a porous medium saturated with pure or saline water at low temperatures, up to 20°C.

Similarity conditions

Following the notation of Gebhart [19], we now define a transformation in terms of a similarity independent variable $\eta(x, y)$ and two similarity dependent variables, $f(\eta)$ and $\phi(\eta)$ as follows:

$$\eta = yb(x); \quad (12)$$

$$f(\eta) = \psi(x,y)/\alpha_1 c(x); \quad (13)$$

$$\phi(\eta) = (t - t_\infty)/(t_0 - t_\infty) \quad (14)$$

where $\psi(x, y)$ is the usual stream function, b and c are transformation functions to be determined to yield a similarity solution. We also define

$$j(x) = t_\infty - t_r, \qquad d(x) = t_0 - t_\infty \quad (15a, b)$$

where t_r is a reference temperature and the surface temperature t_0 may be thought of as greater than the ambient temperature t_∞ for a simpler physical interpretation of the results.

Introducing these transformations into equations (10) and (11) results in

$$f' = \pm K_1 C_1 [|\phi - R|^q - |R|^q], \quad (16)$$

$$\phi'' + C_2 f\phi' - C_3 f'\phi - C_4 f' = 0. \quad (17)$$

The boundary conditions in terms of the similarity variables for a surface at $y = 0$ ($\eta = 0$), are

$$\eta = 0, \quad \phi = 1, \quad f = 0, \quad (18a, b)$$

$$\eta \to \infty, \quad \phi = 0 \quad (19)$$

where

$$K_1 = \frac{\alpha(s,p)K\rho_m g}{\mu\alpha_1}$$

$$C_1 = \frac{d^q}{cb}, \qquad C_2 = \frac{c_x}{b} \quad (20a, b)$$

$$C_3 = \frac{cd_x}{bd}, \qquad C_4 = \frac{cj_x}{bd} \quad (21a, b)$$

and

$$R = \frac{t_m - t_\infty}{t_0 - t_\infty}. \quad (22a)$$

The necessary conditions for which similarity solutions exist are that the quantities C_1, C_2, C_3 and C_4 and the parameter R may be made independent of x. It is worth noting the important role of the quantity R. It places the prescribed temperatures t_0 and t_∞, with respect to $t_m(s, p)$. It also indicates the local direction of the buoyancy force across the thermal region and thus, also the direction of flow. In a recent study of vertical natural convection flows in cold water, Gebhart and Mollendorf [20] have observed local buoyancy reversals for R between 0 and 0.5. A detailed discussion of such reversals in porous layers will be given in the next section.

One may prove that for $C_3 \neq 0$, R may be expressed in terms of C_3 and C_4 as:

$$R = -\frac{j(x)}{d(x)} + \frac{t_m - t_r}{d(x)} = -\frac{C_4}{C_3} + \frac{t_m - t_r + C_5}{d(x)} \quad (22b)$$

where C_5 is a constant of integration. It is then concluded that R is independent of x if

$$t_m - t_r + C_5 = 0.$$

Thus, the reference temperature t_r is given by

$$t_r = t_m + C_5. \quad (23)$$

Considering the equation (16), it is apparent that one approach in seeking possible similarity solutions is to examine particular cases based on the behavior of the parameter R, that is, on the specific variations of the surface temperature t_0 and the ambient temperature t_∞. The four following particular circumstances will be discussed:

1. uniform ambient medium temperature t_∞ and t_0 variable;
2. both t_∞ and t_0 independent of x;
3. t_∞ variable and t_0 constant; and
4. both t_∞ and t_0 variable.

1. Uniform ambient medium temperature t_∞ *and* t_0 *variable*

In this condition $t_0 - t_\infty = d(x)$ is a function of x. Note also that R appears alone in equation (16). Therefore, R [see equation (22)] will be independent of x if and only if t_∞ is equal to t_m. That is

$$R = 0. \quad (24)$$

Then, from equations (20a, b) and (21a), an equation is obtained in terms of c. It may be integrated to give,

$$c_x c^{1-qr} = C_6 \quad (25)$$

where $r = C_3/C_2$ and C_6 is a constant of integration. Exponential or power law variations of c with x result, depending on whether $r = 2/q$ or $r \neq 2/q$.

1(a) *Power-law variation with* x. Integrating equation (25) for $r \neq 2/q$ yields,

$$c(x) = (C_7 x + C_8)^{1/(2-qr)}. \tag{26a}$$

Then, from equations (21a, b) and (22),

$$b(x) = \frac{C_6}{C_2}(C_7 x + C_8)^{(qr-1)/(2-qr)}, \tag{26b}$$

$$d(x) = \left(\frac{C_1 C_6}{C_2}\right)^{1/q}(C_7 x + C_8)^{r/(2-qr)} \tag{26c}$$

where

$$C_7 = C_6(2 - qr),$$

and C_8, the constant of integration, is zero if the origin of the coordinate system is placed at the leading edge for flow.

Similarity solutions. The constants of similarity C_1, C_2, C_3 and C_4 and the constant of integration C_6, are to be calculated. Since there are more unknowns than equations, the constants may be determined such that the similarity equations have a simple and yet general form. The actual bounding temperatures are often known. Therefore, some constants are calculated in terms of these specified conditions. Noting from equation (26c) that the wall-ambient temperature difference varies as a power-law variation with x, it may be assumed that

$$d(x) = Nx^n \tag{27a}$$

where N and n are constants. The expression of b and c become

$$b(x) = \frac{1}{2x}(Ra_x)^{1/2}, \tag{27b}$$

$$c(x) = (Ra_x)^{1/2}, \tag{27c}$$

where Ra_x is a local Rayleigh number defined as

$$Ra_x = \frac{2\alpha(s,p)K\rho_m g d^q x}{\mu\alpha_1} = \rho_m \frac{gxK}{\mu\alpha_1}2\alpha d^q. \tag{28}$$

Equations (16) and (17) then become

$$f' = \pm\,\phi^q, \tag{29}$$

$$\phi'' + (nq+1)f\phi' - 2nf'\phi = 0$$

where $C_1 = 1/K_1$, $C_2 = nq + 1$, and $C_3 = 2n$. Note that the Prandtl number $Pr = \nu/\alpha_1$ does not appear alone in this formulation.

The physical consistency of the above formulation is examined by considering a number of heat transfer quantities. First, the energy convected in the boundary layer at x, $Q(x)$, is expressed as

$$Q(x) = \int_0^\infty c_p\rho u(t - t_\infty)\,\mathrm{d}y \tag{30}$$

$$= \alpha_1 c_p \rho_r c d\int_0^\infty \phi f'\,\mathrm{d}\eta \propto Nx^{[n(2+q)+1]/2}. \tag{31}$$

Second, the thermal boundary layer thickness $\delta(x)$ is obtained from equation (12) as

$$\delta(x) = \eta_\delta/b \propto \eta_\delta x^{(nq-1)/2} \tag{32}$$

where η_δ is a point where ϕ might have a value of 0.01. Combining equations (27b, c) and (31), the convected energy will be constant or increasing with x for $N > 0$, that is, for $t_0 > t_\infty$, if in addition

$$n \geq -\frac{1}{2+q}.$$

Similarly, the thermal boundary layer thickness must be constant or increase with x. This results for $n \leq (1/q)$. Therefore, n must be restricted to the range

$$-\frac{1}{2+q} \leq n \leq \frac{1}{q}. \tag{33}$$

(b) *Exponential variation with* x. For $r = 2/q$, equation (25) is integrated to give,

$$c(x) = C_9\exp(C_6 x). \tag{34a}$$

Then, from equations (20a, b) and (21a), b and d become

$$b(x) = \frac{C_9 C_6}{C_2}\exp(C_6 x), \tag{34b}$$

$$d(x) = \left(\frac{C_1}{C_2}C_9{}^2 C_6\right)^{1/q}\exp\left(\frac{2C_6}{q}x\right) \tag{34c}$$

where C_9 is a constant of integration.

Similarity solutions. From equation (34c), observing the exponential variation with x of the wall–ambient temperature difference, it may be assumed that

$$d(x) = M\exp(mx) \tag{35a}$$

where M and m are constants. Taking C_9 arbitrarily equal to 1, b and c are,

$$b(x) = \frac{m}{2}\exp\left(\frac{mq}{2}x\right), \tag{35b}$$

$$c(x) = \exp\left(\frac{mq}{2}x\right). \tag{35c}$$

The similarity equations given by (16) and (17) are

$$f' = \pm\,\phi^q, \tag{36a}$$

$$\phi'' + qf\phi' - 2f'\phi = 0 \tag{36b}$$

where $C_1 = 1/K_1$, $C_2 = q$, $C_3 = 2$. Again the Prandtl number does not appear. Equations (32) and (35b) indicate that the boundary layer thickness $\delta(x)$ is not zero at $x = 0$. Gebhart and Mollendorf point out the same characteristic in a study of viscous dissipation in external natural convection flows [27]. They conclude that the momentum and energy levels at the leading edge must be small compared to that existing at L, where L is the surface length for this formulation to accurately apply to such a flow downstream of $x = L$.

2. *Both* t_∞ *and* t_0 *independent of* x

From equation (22), R is then independent of x for any values of t_0, t_∞ and t_m. This circumstance is similar to the foregoing power law one, except that n in the equation (27a) is equal to zero, that is, $t_0 - t_\infty = d = N$, a constant. Here R can take any real value. Therefore, proceeding as in the previous case, the following similarity equations may be obtained. For the power-law variation with x, they are

$$f' = \pm[|\phi - R|^q - |R|^q], \quad (37a)$$

$$\phi'' + f\phi' = 0. \quad (37b)$$

It would appear that $t_0 - t_\infty =$ constant might also be accommodated by the exponential form (35a). However, this would require $m = 0$, which is inadmissible in equation (35b, c).

The expressions of b and c obtained in the previous section (equation 27b, c) are still valid. Because of the great practical interest in conditions for which t_0 and t_∞ are constant, numerical solutions to equation (37a, b) have been obtained for a wide range of R. The results are discussed in the next section.

3. t_∞ *is variable and* t_0 *is constant*

If the ambient medium is stratified, the parameter R can be made independent of x if and only if the surface temperature t_0 is equal to the extremum temperature t_m. Thus

$$R = 1. \quad (38)$$

Referring to the expression of the buoyancy force in equation (16), it may be seen that the resulting flow is always down. Before solving equations (20) and (21), it should be noted from equation (21b), that the form of $j(x)$ depends entirely on those of b, c and d, which have already been determined in equations (27a, b, c) and (35a, b, c).

3(a) *Power-law variation of* t_∞ *with* x. From equations (21b) and (27b, c), it may be shown that,

$$C_4 = -2n.$$

Then, $j(x)$ becomes

$$j(x) = -Nx^n + C_{10} \quad (39)$$

where $C_{10} = t_0 - t_m - C_5$. The similarity equations are

$$f' - (1-\phi)^q + 1 = 0, \quad (40a)$$

$$\phi'' + (nq+1)f\phi' - 2nf'(\phi - 1) = 0. \quad (40b)$$

For any solution to be realistic, any stratification of the ambient medium must be stable, since it is assumed quiescent in the boundary conditions (8a, b).

A rough measure of ambient medium stability is that ρ_∞ must be constant or decreasing upward, independent of the sign of $t_0 - t_\infty$. That is, $(d\rho_\infty/dx) \le 0$ for upflow and $(d\rho_\infty/dx) \ge 0$ for downflow. See Fig. 1. Now,

$$\frac{d\rho_\infty}{dx} \simeq \frac{d\rho_\infty}{dt_\infty}\frac{dt_\infty}{dx} = \alpha q(t_m - t_\infty)^{q-1} j_x. \quad (41)$$

Thus, $\rho_{\infty x}$ is greater or equal to zero for j_x greater or equal to zero. Therefore, from equations (33) and (39), the requirement that the environment is stable is that n is restricted to:

$$-\frac{1}{2+q} \le n \le 0. \quad (42)$$

(b) *Exponential variation of* t_∞ *with* x. From $C_3 = 2$ and from equation (22b), $C_4 = -2$. Then $j(x)$ is expressed as

$$j(x) = -M\exp(mx), \quad (43)$$

and the similarity equations as

$$f' - (1-\phi)^q + 1 = 0, \quad (44a)$$

$$\phi'' + qf\phi' - 2f'(\phi - 1) = 0. \quad (44b)$$

The remarks on the thermal boundary-layer thickness in part 1(b) also hold for this circumstance.

4. *Both* t_∞ *and* t_0 *variable*

This circumstance is similar to the one wherein the temperature t_0 at $y = 0$ varies with x while the ambient temperature t_∞ is constant. Here, solutions are limited by the permissible values of R. Also, the solutions must be restricted to those which result in a stable ambient medium. The solutions of this section, being the most general, also include the previous three circumstances. The definition of b and c will here remain the same.

(*a*) *Power-law variation with* x. Noticing from equation (21b) that the variation of $j(x)$ with respect to x depends on the sign of C_4, let

$$C_4 = 2nC_4'.$$

Therefore,

$$j(x) = C_4' N x^n + C_{11} \quad (45)$$

where C_{11} is a constant of integration that can be taken equal to zero since the reference temperature t_r is defined within an arbitrary constant C_5. From equations (22) and (30), C_4' becomes

$$C_4' = -R. \quad (46)$$

$j(x)$ is rewritten as

$$j(x) = -RNx^n.$$

Since the value of R determines the direction of the flow, the range of values of n will depend on it. The similarity equations are given by

$$f' = \pm[|\phi - R|^q - |R|^q] \quad (47a)$$

$$\phi'' + (nq+1)f\phi' - 2nf'(\phi - R) = 0. \quad (47b)$$

(b) *Exponential variation with* x. The stability of the medium depends on the constant of similarity C_4. Taking arbitrarily $C_3 = 2$ and from equation (22b), $C_4 = -2R$, $j(x)$ can be expressed as

$$j(x) = -RM\exp(mx).$$

The similarity equations are

$$f' = \pm[|\phi - R|^q - |R|^q], \quad (48a)$$

$$\phi'' + qf\phi' - 2f'(\phi - R) = 0. \quad (48b)$$

3. NUMERICAL CALCULATIONS FOR PURE WATER

Numerical solutions have been obtained for equation (37a, b) subject to boundary conditions given in equations (18) and (19), rewritten below. These equations describe the circumstance of natural convection adjacent to an isothermal surface embedded in unstratified saturated porous media.

$$f' = \pm [|\phi - R|^q - |R|^q], \tag{37a}$$

$$\phi'' + f\phi' = 0, \tag{37b}$$

$$\eta = 0, \quad \phi = 1, \quad f = 0, \tag{18}$$

$$\eta \to \infty, \quad \phi = 0. \tag{19}$$

Before discussing the computed results, the basic transport quantities of interest are determined, in terms of similarity variables, as follows:

$$u(x, y) = \alpha_1 \frac{Ra_x f'}{2x},$$

$$-v(x, y) = \alpha_1 \frac{(Ra_x)^{1/2}}{2x}(f - \eta f'),$$

$$Nu_x = \frac{q'' x}{k(t_0 - t_\infty)} = \left[\frac{-\phi'(0)}{2}\right](Ra_x)^{1/2}$$

where

$$\eta = \frac{1}{2}(Ra_x)^{1/2}\frac{y}{x}$$

where q'' is the surface heat flux and Nu_x is the local Nusselt number. The buoyancy force **F** is also of importance in determining the direction of the flow and, therefore, the choice of the coordinate system.

$$F = g(\rho_\infty - \rho).$$

In terms of the similarity variables this becomes

$$F = \alpha(s, p)\rho_m g d^q [|\phi - R|^q - |R|^q]. \tag{48}$$

From equation (37a), the tangential velocity is seen to be directly proportional to the buoyancy force. Therefore, local flow reversal occurs across the convective layer where F changes sign. The point where this occurs is called η_c. It is determined from:

$$F = 0 = \alpha(s, p)\rho_m d^q [|\phi(\eta_c) - R|^q - |R|^q].$$

The only reasonable solution of this equation is

$$\phi(\eta_c) = 2R. \tag{49}$$

Thus, for any given and admissible value of R, there may be a location η_c, where the buoyancy force and tangential velocity are zero. The flow has opposite directions on opposite sides of this location in the thermal layer. Since the temperature distribution ϕ decreases monotonically from 1 to 0, the values of R for which buoyancy and flow reversals will occur are in the range $1/2 \geq R \geq 0$, from equation (49). For $R = 1/2$, $F = 0$ at the surface, as shown in Fig. 5. Further, for $R \leq 0$, F is upward and so is the flow. For $R \geq 1/2$, both are downward.

The occurrence of a local buoyancy-force reversal made the numerical convergence extremely slow on approaching $R = 0.194$ from $R = 0$, and on approaching $R = 0.401$ from $R = 0.5$. Convergence was not obtained for values of R in the range $0.195 \leq R \leq 0.40$. In a study of natural convection about a vertical flat surface in cold water, Carey, Gebhart and Mollendorf [21] had similar difficulties and could not obtain

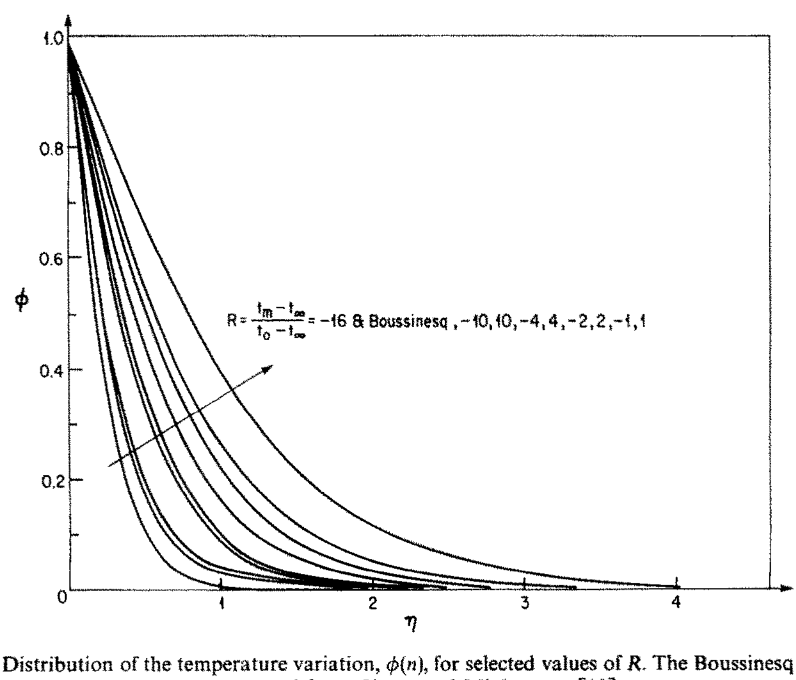

FIG. 2. Distribution of the temperature variation, $\phi(\eta)$, for selected values of R. The Boussinesq results are computed from Cheng and Minkowycz [13].

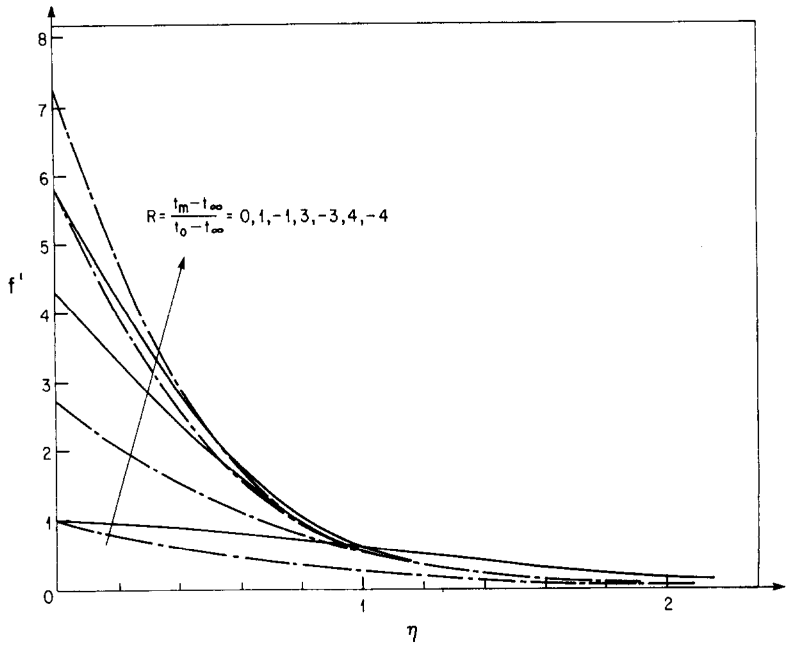

FIG. 3. Distribution of the tangential component of velocity, f', for selected values of R outside the buoyancy-reversal region. The solid lines are for downward flow, the broken lines for upward flow.

convergence for R in the range $0.15 < R < 0.29$. Local flow reversal at the surface first occurred for R near 0.30.

The boundary-layer approximations are questionable in bi-directional flows. These approximations rely on the condition that the changes in the physical quantities are small, with respect to x, compared to their changes with respect to y. Thus, $\partial u/\partial y$ is assumed large compared to $\partial v/\partial x$, to result in equation (9). From an order of magnitude estimate, it may be shown that

$$u/v = O(Ra_x)^{1/2},$$

and

$$y/x = O(Ra_x)^{-1/2}.$$

That is, the solutions are more reliable at increasing Ra_x. Note that the Rayleigh number may be very large for any value of R even in the range 0–1/2.

4. RESULTS

The equations (18), (19) and (36a, b) were integrated numerically by the Runge–Kutta method. The value of $\phi'(0)$ was successively approximated by a shooting technique, for different values of R. Computations were first performed for $R = \pm 10$, ± 4, ± 2, ± 3, ± 1, ± 0.5 and 0, for $q(s,p) = q(0,1) = 1.894816$. Resulting temperature distributions for $R = -16$, ± 10, ± 4, ± 2 and ± 1 are plotted in Fig. 2. The variation of the tangential component of the filtration velocity is shown in Fig. 3 for selected values of R. The normal velocity component is plotted in Fig. 4. In both Figs. 3 and 4, the dashed curves represent upward flow.

Then, the region where buoyancy-force reversals occur was investigated. Results were obtained for $0 \le R \le 0.194$ and $0.401 \le R \le 0.5$. These values of R correspond to small flow reversals. Instability of the numerical routine resulted in the range $0.195 < R < 0.40$, a region of large reversals. Convective inversion occurred in this range. In Fig. 5, the vertical velocity component is plotted for selected values of R across the range of local flow reversal. For $R = 0.45$, the reversal is seen to occur at $\eta_c = 0.40$. Referring to Fig. 6, where temperature distribution for different values of R is plotted, $\phi(\eta_c = 0.4)$ is seen to be equal to $\phi(\eta_c) \simeq 0.90 = 2R$. Figure 7 shows the very large variation of the heat transfer over the whole range of R for $q(s,p) = q(0,1)$ and for $q(0,500)$. There is a large decrease in heat transfer as the buoyancy-force reversal region is approached from each side. However, the effects of convective inversion on the form of the temperature distributions appear to be relatively small.

Acknowledgements — Mr. J. Ramilison acknowledges support for his studies and research from the African–American Institute. The second author wishes to acknowledge support for this research under National Science Foundation Grant ENG 77-21641.

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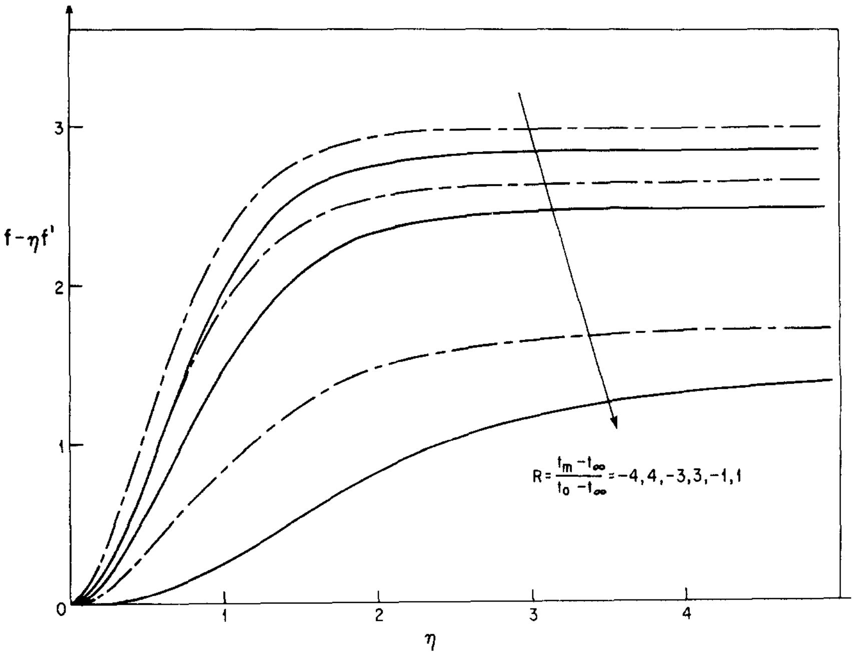

FIG. 4. Distribution of the normal velocity component for selected R outside the buoyancy-reversal region. Solid lines represent downward flow and the broken lines, upward flow.

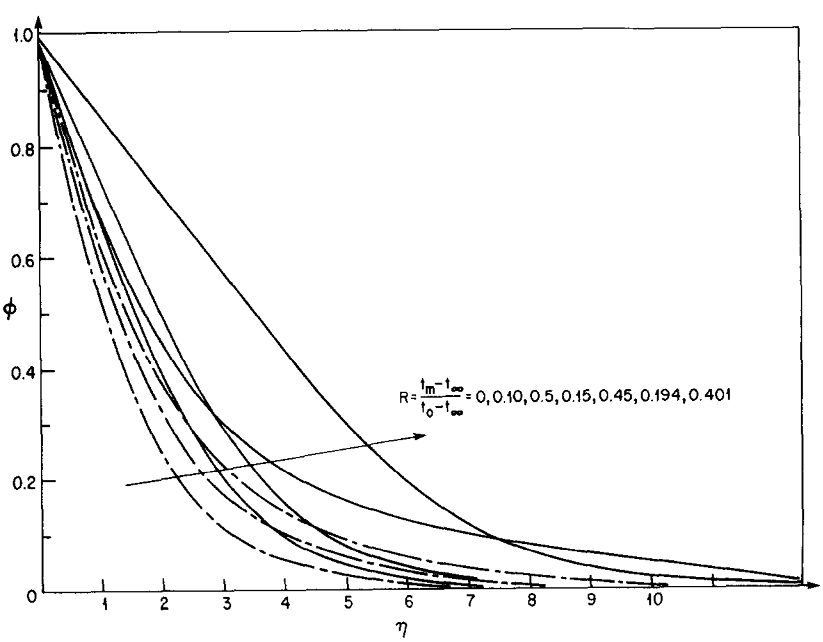

FIG. 5. Calculated distribution of the velocity component parallel to the surface for selected R inside the buoyancy-reversal region. Solid curves are for downward flow, broken curves for upward flow.

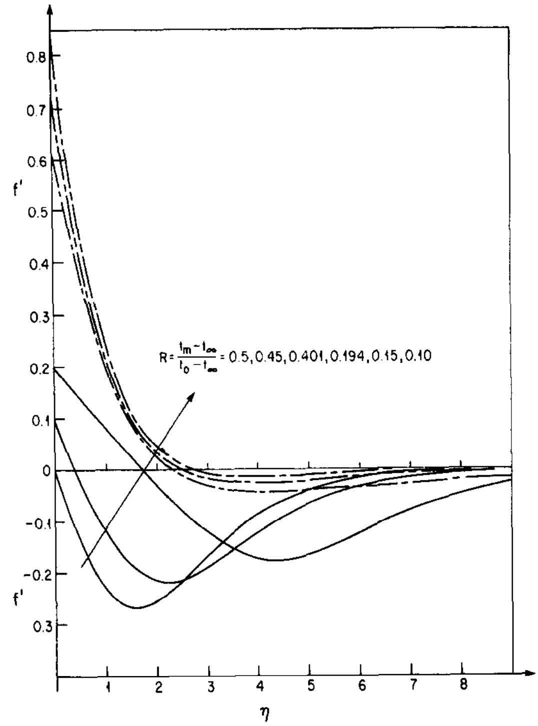

FIG. 6. Calculated temperature distribution for selected values of R in the buoyancy-reversal region.

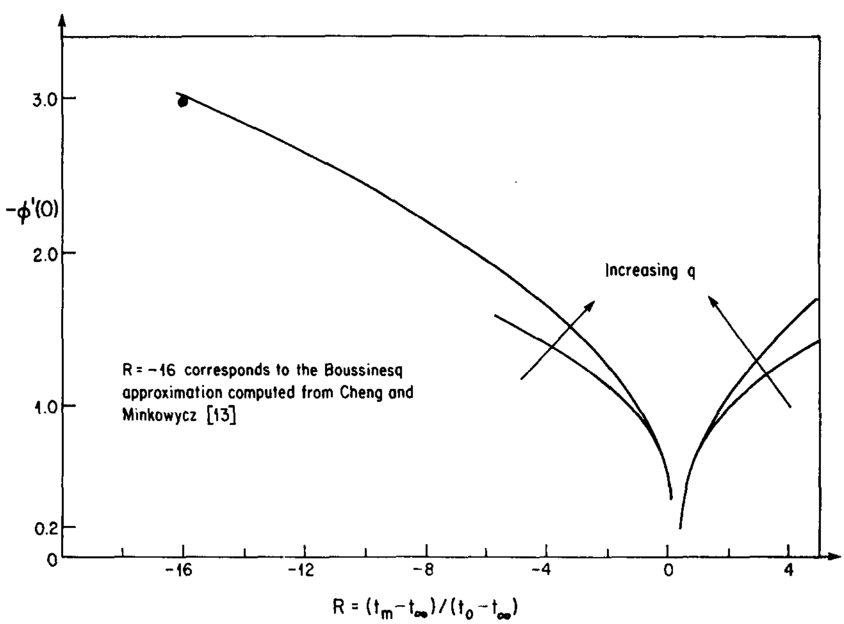

FIG. 7. Heat transfer dependence on R for $(g(s,p) = q(0,1) = 1.894816$ and $q(0,500) = 1.727147$. The arrow indicates increasing q.

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TRANSPORT PAR GRAVITE DANS DES MILLIEUX POREUX SATURES D'EAU PURE OU SALINE A TEMPERATURE BASSE

Résumé—L'eau terrestre se trouve fréquemment à des niveaux de température tels qu'apparait l'influence de l'extremum de densité. L'effet d'Archimède devient alors compliqué, avec un renversement local de l'écoulement et une inversion de convection. Les différences de densité ne peuvent pas s'exprimer linéairement en fonction de la température. On étudie de tels écoulements à partir d'une équation d'état pour la densité qui est d'une grande précision, bien que de forme simple. Les équations de mouvement sont appliquées à des écoulements plans verticaux dans un milieu poreux. De nombreux paramètres apparaissent. Néanmoins, une formulation concerne plusieurs applications intéressantes et pratiques pour une grande diversité de conditions de température. Des solutions spécifiques sont données pour l'écoulement et le transport avec des conditions aux limites de température telles qu'apparaissent le renversement des forces d'Archimède et l'inversion de la convection.

TRANSPORT DURCH AUFTRIEB IN MIT REINEM ODER SALZHALTIGEN WASSER GESÄTTIGTEN PORÖSEN MEDIEN BEI NIEDRIGEN TEMPERATUREN

Zusammenfassung—Das Wasser auf der Erdoberfläche hat gewöhnlich Temperaturen, bei denen ein Dichteextremum oder der Trend zu einem Extremum vorliegt. Die Abhängigkeit der Auftriebskraft von der Temperatur kann in diesem Fall sehr kompliziert werden, wobei lokale Strömungsumkehr und konvektive Inversion auftreten können. Dichteunterschiede können dann nicht mehr als lineare Funktion der Temperatur ausgedrückt werden. Strömungen unter solchen Bedingungen werden hier behandelt, wobei die Auftriebskraft aus einer Zustandsgleichung für die Dichte berechnet wird, die sehr genau, aber trotzdem recht einfach ist. Die erhaltenen Gleichungen für den Transport werden auf vertikale ebene Strömungen in einem porösen Medium angewandt. Dabei treten mehrere neue Parameter auf. Es ergibt sich jedoch eine Ähnlichkeitslösung für viele interessierende praktische Anwendungsfälle für eine große Vielfalt von Temperaturbedingungen. Spezielle Lösungen für Strömung und Transport werden für Temperaturrandbedingungen angegeben, bei denen lokale Umkehr der Auftriebskraft sowie konvektive Inversion auftreten.

СВОБОДНОКОНВЕКТИВНОЕ ДВИЖЕНИЕ ЖИДКОСТИ В ПОРИСТЫХ СРЕДАХ, ПРОПИТАННЫХ ЧИСТОЙ И СОЛЕНОЙ ВОДОЙ, ПРИ НИЗКИХ ТЕМПЕРАТУРАХ

Аннотация — В земных условиях вода зачастую находится при такой температуре, когда наблюдается экстремум или тенденция к экстремуму плотности. При этом действие подъёмной силы может носит очень сложный характер и сопровождаться локальным реверсированием потока и конвективной инверсией. В этом случае разность плотностей нельзя описать линейной температурной зависимостью. В статье дается формулировка уравнений для таких потоков. Величина подъёмной силы очень точно и в то же время довольно просто рассчитывается из уравнения состояния для плотности. С помощью полученных уравнений можно рассчитывать вертикальное двумерное течение жидкости в пористой среде. При этом появляется несколько новых параметров. Однако результаты, полученные с помощью теории подобия, могут использоваться на практике в широком диапазоне температур. Даны частные решения для случаев течения и переноса массы при определенных значениях температуры, при которых наблюдаются локальное изменение направления действия подъёмной силы и конвективная инверсия.